Maths · Society

Notebook

Tetration, Pentration and Hyperoperations

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Hyperoperations sequence

The Hyperoperation sequence is an infinite sequence of operations, with the next in the sequence allowing for larger numbers to be made. Simple operations like addition, multiplication, and exponents are the first elements in this sequence, but it continues!

Meet Tetration, also known as Hyper-4. Similar to how multiplication is repeated addition, or exponents are repeated multiplication, tetration is repeated exponentiation.

To understand tetration, you first need to understand the basic concept that underlies it. In the more common Rudy Rucker Notation, tetration is written in the format ^{n}x where n refers to the number of ‘floors’ used in the exponent.

A basic exponent, like 2^2 is 2 floors high. So, an exponent like $2^{2^\wedge 2}$, has 3 floors.

To solve these exponents, you should work from the highest 2 floors down. So, if you wanted to do $2^{2^\wedge 2}$ first you solve 2^2 and then 2^4, solving from the inside outwards.

There is one key point with Tetration notation. Every floor must have x as the power, or else it can’t be expressed as Tetration, and would instead use basic exponential notation.

If Tetration is repeated exponentiation, then is it possible for there to be repeated Tetration?

The answer is Yes. The next step is Pentation, Hyper-5, tetration’s big brother.

Did you know?

You might have noticed a key reason you don’t hear about tetration very often. Using it creates large numbers very quickly…

$^{3}10$ is 1 with 10 billion zeros, which is larger than the number of atoms in the observable universe. If anyone wanted to write this number on an infinite supply of paper, they would be dead before they could do it.

Pentation is written in multiple formats, and there isn’t a common standardised one like for tetration, instead we’ll use the most simple format, which looks similar to tetration.

[2 pentated 3 times] is an example of a number in this format. It might not look that complex, but this number has 3 floors of tetration, which looks like this $^{2^\wedge 2}2$

This might seem simple, but it isn’t; the solution is 65,536!

So can pentation be repeated? Of course it can! Hexation is the start of the more complex areas in the Hyperoperation sequence and can’t be written in a simple format like Tetration or Pentation. It is, put simply, repeated Pentation and doesn’t have many real-world use cases.

In Theory, the hyperoperation sequence can continuously recur, and would instead begin to use a formula and formal calculations to solve, instead of the simple “floor” concept and simpler notation from before. The formula for this is shown on below, but doesn’t account for addition. n represents the operation being conducted. For multiplication, n=2, for exponentiation, n=3, for Tetration, n=4, and for Pentation n=5, etc.

$$a[n]b = \underbrace{a[n-1](a[n-1](a[n-1](\cdots[n-1](a[n-1](a[n-1]a))\cdots)))}_{b \text{ copies of } a}, \quad n \geq 2$$

So why are these Hyperoperations useful? Some functions and concepts can be extended using hyperoperation. They are essential to large numbers comprehension. New larger infinities and their properties arise using its ideal notion and class notion, and derivative and integral concepts from calculus are also extended with hyperoperations.

To summarise, Hyperoperations are a complex and fascinating sequence, showing how much maths lies beyond the day-to-day operations we use. ■

The History

One of the earliest mentions of hyperoperations was that of Albert Bennett in 1914, in which he developed part of his theory of commutative hyperoperations. Twelve years later, Wilhelm Ackermann defined ‘ϕ,’ a function like a sequence of hyperoperations. In a 1947 paper, the Greek Ruben Goodstein introduced a sequence of operations now called hyperoperations, and extended operations beyond exponentiation, adding the next steps, suggesting they be called tetration and pentation. As a function with three arguments, it is seen to be a version of the original Ackermann function.